

Contents

1	Introduction	1
Part I Looking over Escher's Shoulder		
2	Symmetries and Fundamental Domains	9
2.1	What is Symmetry?	9
2.2	What Movements are There?	11
2.3	Groups of Movements	16
2.4	Discontinuous Groups and Fundamental Domains	21
3	The Discontinuous Symmetry Groups of the Plane	25
3.1	How Many Different Groups of Movements are There?	25
3.2	Finite Groups of Movements	31
3.3	The Subgroup of Translations	37
3.4	The 7 Frieze Groups	40
3.4.1	$\mathcal{F}_1$: Only Translations (<i>nnnn</i>)	43
3.4.2	$\mathcal{F}_1^1$: Only Reflections of Type 1 (<i>jnnn</i>)	44
3.4.3	$\mathcal{F}_1^2$: Only Reflections of Type 2 (<i>njnn</i>)	45
3.4.4	$\mathcal{F}_1^3$: Proper Glide Reflections (<i>nnnj</i>)	46
3.4.5	$\mathcal{F}_2$: Only Rotations (<i>nnjn</i>)	47
3.4.6	$\mathcal{F}_2^1$: Rotations, Type-1 and Type-2 Reflections (<i>jjjn</i>)	48
3.4.7	$\mathcal{F}_2^2$: Proper Glide Reflections, Type-2 Reflections, and Rotations (<i>njjj</i>)	50
3.4.8	Summary	51
3.4.9	Classification: A Test	51
3.4.10	Hints for Artists	53
3.5	The 17 Plane Crystal Groups	53
3.5.1	The Crystallographic Restriction	54
3.5.2	Translations, Reflections: 4 Groups	56
3.5.3	Translations, 2-Rotations, Reflections: 5 Groups	68
3.5.4	Translations, 3-Rotations, (Glide) Reflections: 3 Groups	76

3.5.5	Translations, 4-Rotations, Reflections: 3 Groups	84
3.5.6	Translations, 6-Rotations, Reflections: 2 Groups	88
3.5.7	Classification: A Test	91
4	The Heesch Constructions	101
4.1	Lattices and Nets	102
4.2	The Heesch Construction: Motivation	109
4.3	The Heesch Constructions: 28 Methods	112

References for Part I

Part II Möbius Transformations

5	Möbius Transformations	145
5.1	Complex Numbers: Some Reminders	145
5.2	Möbius Transformations: Definitions and First Results	147
5.3	Möbius Transformations and Circles	150
5.4	Fixed Points of Möbius Transformations	156
5.5	Conjugate Möbius Transformations	158
5.6	Characterization: Fixed Points in $\{0, \infty\}$	160
5.7	Characterization: the General Case	167
5.8	Wish List/Visualization	173
6	Groups of Möbius Transformations	177
6.1	First Examples of Groups of Möbius Transformations	177
6.2	Fundamental Domains and Discrete Groups	179
6.3	Special Möbius Transformations	182
6.4	Digression: Hyperbolic Geometry	191
6.4.1	Hyperbolic Geometry I: The Upper Half-plane $\mathbb{H}$	192
6.4.2	Hyperbolic Geometry II: The Unit Circle $\mathbb{U}$	198
6.5	The Modular Group	199
6.6	Groups with Two Generators	204
6.7	Schottky Groups	206
6.8	The Mystery of the Parabolic Commutator	218
6.9	The Structure of Kleinian Groups	226
6.9.1	The Isometric Circles	226
6.9.2	The Limit Set	230
6.9.3	A Fundamental Domain	233
6.10	Parabolic Commutators: Construction	238

References for Part II

Part III Penrose Tilings

7 Penrose Tilings. 249

7.1 Non-periodic Tilings: The Problem 250

7.2 The “Golden” Penrose Triangles 253

7.3 Which Tiling Patterns are Possible? 257

7.4 Index Sequences Generate Tilings. 262

7.5 Isomorphisms of Penrose Tilings. 272

7.6 Supplements. 275

References for Part III

Index. 281