

# **Numerics of Unilateral Contacts and Friction**

**Modeling and Numerical Time Integration in  
Non-Smooth Dynamics**

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