

# Contents

|  |    |
|--|----|
| <b>Preface</b>   | v  |
| <b>1 Topological Groups</b>                                      | 1  |
| 1.1 The Notion of a Topological Group . . . . .                  | 1  |
| 1.2 The Neighborhood Filter of the Identity . . . . .            | 4  |
| 1.3 The Topology $\mathcal{T}(\mathcal{F})$ . . . . .            | 7  |
| 1.4 Topologizing a Group . . . . .                               | 10 |
| 1.5 Metrizable Refinements . . . . .                             | 14 |
| 1.6 Topologizability of a Countably Infinite Ring . . . . .      | 18 |
| <b>2 Ultrafilters</b>  | 26 |
| 2.1 The Notion of an Ultrafilter . . . . .                       | 26 |
| 2.2 The Space $\beta D$ . . . . .                                | 29 |
| 2.3 Martin's Axiom . . . . .                                     | 34 |
| 2.4 Ramsey Ultrafilters and $P$ -points . . . . .                | 36 |
| 2.5 Measurable Cardinals . . . . .                               | 39 |
| <b>3 Topological Spaces with Extremal Properties</b>             | 41 |
| 3.1 Filters and Ultrafilters on Topological Spaces . . . . .     | 41 |
| 3.2 Spaces with Extremal Properties . . . . .                    | 42 |
| 3.3 Irresolvability . . . . .                                    | 46 |
| <b>4 Left Invariant Topologies and Strongly Discrete Filters</b> | 52 |
| 4.1 Left Topological Semigroups . . . . .                        | 52 |
| 4.2 The Topology $\mathcal{T}[\mathcal{F}]$ . . . . .            | 54 |
| 4.3 Strongly Discrete Filters . . . . .                          | 56 |
| 4.4 Invariant Topologies . . . . .                               | 63 |
| <b>5 Topological Groups with Extremal Properties</b>             | 68 |
| 5.1 Extremally Disconnected Topological Groups . . . . .         | 68 |
| 5.2 Maximal Topological Groups . . . . .                         | 71 |
| 5.3 Nodec Topological Groups . . . . .                           | 72 |
| 5.4 $P$ -point Theorems . . . . .                                | 76 |
| <b>6 The Semigroup <math>\beta S</math></b>                      | 82 |
| 6.1 Extending the Operation to $\beta S$ . . . . .               | 82 |
| 6.2 Compact Right Topological Semigroups . . . . .               | 86 |

|           |  |     |
|-----------|--|-----|
| 6.3       | Hindman's Theorem . . . . .                                      | 91  |
| 6.4       | Ultrafilters from $K(\beta S)$ . . . . .                         | 94  |
| <b>7</b>  | <b>Ultrafilter Semigroups</b>                                    | 97  |
| 7.1       | The Semigroup $\text{Ult}(\mathcal{T})$ . . . . .                | 97  |
| 7.2       | Regularity . . . . .   | 101 |
| 7.3       | Homomorphisms . . . . .  | 105 |
| <b>8</b>  | <b>Finite Groups in <math>\beta G</math></b>                     | 110 |
| 8.1       | Local Left Groups and Local Homomorphisms . . . . .              | 110 |
| 8.2       | Triviality of Finite Groups in $\beta \mathbb{Z}$ . . . . .      | 117 |
| 8.3       | Local Automorphisms of Finite Order . . . . .                    | 120 |
| 8.4       | Finite Groups in $G^*$ . . . . .                                 | 128 |
| <b>9</b>  | <b>Ideal Structure of <math>\beta G</math></b>                   | 130 |
| 9.1       | Left Ideals . . . . .  | 130 |
| 9.2       | Right Ideals . . . . .   | 136 |
| 9.3       | The Structure Group of $K(\beta G)$ . . . . .                    | 140 |
| 9.4       | $K(\beta G)$ is not Closed . . . . .                             | 144 |
| <b>10</b> | <b>Almost Maximal Topological Groups</b>                         | 147 |
| 10.1      | Construction . . . . .   | 147 |
| 10.2      | Properties . . . . .   | 150 |
| 10.3      | Semilattice Decompositions and Burnside Semigroups . . . . .     | 154 |
| 10.4      | Projectives . . . . .  | 158 |
| 10.5      | Topological Invariantness of $\text{Ult}(\mathcal{T})$ . . . . . | 165 |
| <b>11</b> | <b>Almost Maximal Spaces</b>                                     | 170 |
| 11.1      | Right Maximal Idempotents in $\mathbb{H}_\kappa$ . . . . .       | 170 |
| 11.2      | Projectivity of $\text{Ult}(\mathcal{T})$ . . . . .              | 174 |
| 11.3      | The Semigroup $C(p)$ . . . . .                                   | 177 |
| 11.4      | Local Monomorphisms . . . . .                                    | 181 |
| <b>12</b> | <b>Resolvability</b>   | 188 |
| 12.1      | Regular Homeomorphisms of Finite Order . . . . .                 | 188 |
| 12.2      | Resolvability of Topological Groups . . . . .                    | 194 |
| 12.3      | Absolute Resolvability . . . . .                                 | 198 |
| <b>13</b> | <b>Open Problems</b>   | 208 |
|           | <b>Bibliography</b>  | 211 |
|           | <b>Index</b>   | 217 |