

Contents

1	Introduction and Overview	1
1.1	Objectives	1
1.2	Past and Present	7
1.3	Methods	11
1.4	Supplements	14
1.5	Arrangement of Contents	15
2	Linear Wave Equations	17
2.1	Expression of Solutions	17
2.1.1	Expression of Solutions When $n \leq 3$	19
2.1.2	Method of Spherical Means	20
2.1.3	Expression of Solutions When $n(> 1)$ Is Odd	25
2.1.4	Expression of Solutions When $n(\geq 2)$ Is Even	26
2.2	Expression of Fundamental Solutions	28
2.3	Fourier Transform	31
2.4	Appendix—The Area of Unit Sphere	32
3	Sobolev Type Inequalities with Decay Factor	35
3.1	Preliminaries	35
3.1.1	Commutant Relations	36
3.1.2	$L^{p,q}(IR^n)$ space	38
3.1.3	Generalized Sobolev Norms	39
3.1.4	Commutativity with the Wave Operator	41
3.1.5	Representing Derivatives Under Ordinary Coordinates by Derivatives Under Polar Coordinates	42
3.2	Some Variations of Classical Sobolev Embedding Theorems	45
3.2.1	Sobolev Embedding Theorems on a Unit Sphere	45
3.2.2	Sobolev Embedding Theorems on a Ball	46

3.2.3	Sobolev Embedding Theorems on an Annulus	47
3.2.4	Sobolev Embedding Theorems for Decomposed Dimensions	49
3.3	Sobolev Embedding Theorems Based on Binary Partition of Unity	51
3.3.1	Binary Partition of Unity	51
3.3.2	Sobolev Embedding Theorems Based on Binary partition of Unity	53
3.4	Sobolev Type Inequalities with Decay Factor	58
3.4.1	Sobolev Type Inequalities with Decay Factor Inside the Characteristic Cone	58
3.4.2	Sobolev Type Inequalities with Decay Factor on the Entire Space	61
4	Estimates on Solutions to the Linear Wave Equations	65
4.1	Estimates on Solutions to the One-Dimensional Linear Wave Equations	65
4.2	Generalized Huygens Principle	67
4.3	Estimates on Solutions to the Two-Dimensional Linear Wave Equations	69
4.4	An L^2 Estimate on Solutions to the $n(\geq 4)$ -Dimensional Linear Wave Equations	72
4.5	$L^{p,q}$ Estimates on Solutions to the Linear Wave Equations	79
4.6	L^1-L^∞ Estimates on Solutions to the Linear Wave Equations	85
4.6.1	L^1-L^∞ Estimates on Solutions to the Homogeneous Linear Wave Equation	85
4.6.2	L^1-L^∞ Estimates on Solutions to the Inhomogeneous Linear Wave Equations	91
4.6.3	L^1-L^∞ Estimates on Solutions to the Linear Wave Equations	100
5	Some Estimates on Product Functions and Composite Functions	113
5.1	Some Estimates on Product Functions	113
5.2	Some Estimates on Composite Functions	121
5.3	Appendix—A Supplement About the Estimates on Product Functions	129
6	Cauchy Problem of the Second-Order Linear Hyperbolic Equations	133
6.1	Introduction	133

6.2	Existence and Uniqueness of Solutions	134
6.3	Regularity of Solutions	147
7	Reduction of Nonlinear Wave Equations to a Second-Order Quasi-linear Hyperbolic System	155
7.1	Introduction	155
7.2	Case of a General Nonlinear Right-Hand Side Term F	157
7.3	Cases of Special Nonlinear Right-Hand Side Terms F	159
8	Cauchy Problem of One-Dimensional Nonlinear Wave Equations	161
8.1	Introduction	161
8.2	Lower Bound Estimates on the Life-Span of Classical Solutions to Cauchy Problem (8.1.14)–(8.1.15)	163
8.2.1	Metric Space $X_{S,E,T}$. Main results	163
8.2.2	Framework to Prove Theorem 8.2.1—The Global Iteration Method	166
8.2.3	Proof of Lemma 8.2.5	170
8.2.4	Proof of Lemma 8.2.6	173
8.3	Lower Bound Estimates on the Life-Span of Classical Solutions to Cauchy Problem (8.1.14)–(8.1.15) (Continued)	175
8.3.1	Metric Space $X_{S,E,T}$. Main results	175
8.3.2	Proof of Lemma 8.3.1	176
8.3.3	Proof of Lemma 8.3.2	180
9	Cauchy Problem of $n(\geq 3)$-Dimensional Nonlinear Wave Equations	183
9.1	Introduction	183
9.2	Lower Bound Estimates on the Life-Span of Classical Solutions to Cauchy Problem (9.1.11)–(9.1.12)	186
9.2.1	Metric Space $X_{S,E,T}$. Main Results	186
9.2.2	Framework to Prove Theorem 9.2.1—The Global Iteration Method	188
9.2.3	Proof of Lemma 9.2.5	191
9.2.4	Proof of Lemma 9.2.6	196
9.2.5	The Case that the Nonlinear Term on the Right-Hand Side Does not Depend on u Explicitly: $F = F(DU, D_x DU)$	200
9.3	Lower Bound Estimates on the Life-Span of Classical Solutions to Cauchy Problem (9.1.11)–(9.1.12) (Continued)	201
9.3.1	Metric Space $X_{S,E,T}$. Main results	202
9.3.2	Framework to Prove Theorem 9.3.1—The Global Iteration Method	203

9.3.3	Proof of Lemma 9.3.5	206
9.3.4	Proof of Lemma 9.3.6	215
10	Cauchy Problem of Two-Dimensional Nonlinear Wave Equations	217
10.1	Introduction	217
10.2	Lower Bound Estimates on the Life-Span of Classical Solutions to Cauchy Problem (10.1.14)–(10.1.15) (The Case $\alpha = 1$)	220
10.2.1	Metric Space $X_{S,E,T}$. Main Results	220
10.2.2	Framework to Prove Theorem 10.2.1—The Global Iteration Method.	222
10.2.3	Proof of Lemmas 10.2.5 and 10.2.6.	224
10.3	Lower Bound Estimates on the Life-Span of Classical Solutions to Cauchy Problem (10.1.14)–(10.1.15) (The Case $\alpha \geq 2$).	232
10.3.1	Metric Space $X_{S,E,T}$. Main Results	232
10.3.2	Framework to Prove Theorem 10.3.1—The Global Iteration Method.	234
10.3.3	Proof of Lemmas 10.3.3 and 10.3.4.	234
10.4	Lower Bound Estimates on the Life-Span of Classical Solutions to Cauchy Problem (10.1.14)–(10.1.15) (The Cases $\alpha = 1$ and 2) (Continued)	241
10.4.1	Metric Space $X_{S,E,T}$. Main Results	242
10.4.2	Framework to Prove Theorem 10.4.1—The Global Iteration Method.	243
10.4.3	Proof of Lemmas 10.4.3 and 10.4.4.	243
11	Cauchy Problem of Four-Dimensional Nonlinear Wave Equations	255
11.1	Introduction	255
11.2	Lower Bound Estimates on the Life-Span of Classical Solutions to Cauchy Problem (11.1.11)–(11.1.12)	257
11.2.1	Metric Space $X_{S,E,T}$. Main Results	257
11.2.2	Framework to Prove Theorem 11.2.1—The Global Iteration Method.	258
11.2.3	Proof of Lemmas 11.2.5 and 11.2.6.	260
12	Null Condition and Global Classical Solutions to the Cauchy Problem of Nonlinear Wave Equations	263
12.1	Introduction	263
12.2	Null Condition and Global Existence of Classical Solutions to the Cauchy Problem of Three-Dimensional Nonlinear Wave Equations	265

12.2.1	Null Condition of Three-Dimensional Nonlinear Wave Equations	265
12.2.2	Some Properties of the Null Form	273
12.2.3	Metric Space $X_{S,E}$. Main Results	277
12.2.4	Proof of Lemmas 12.2.4 and 12.2.5	280
12.3	Null Condition and Global Existence of Classical Solutions to the Cauchy Problem of Two-Dimensional Nonlinear Wave Equations	286
12.3.1	Introduction	286
12.3.2	Metric Space $X_{S,E}$. Main Results	290
12.3.3	Proof of Lemmas 12.3.1 and 12.3.2	291
13	Sharpness of Lower Bound Estimates on the Life-Span of Classical Solutions to the Cauchy Problem—The Case that the Nonlinear Term $F = F(Du, D_x Du)$ on the Right-Hand Side Does not Depend on u Explicitly	303
13.1	Introduction	303
13.2	Upper Bound Estimates on the Life-Span of Classical Solutions to the Cauchy Problem of a Kind of Semi-linear Wave Equations	305
13.3	Proof of the Main Results	312
14	Sharpness of Lower Bound Estimates on the Life-Span of Classical Solutions to the Cauchy Problem—The Case that the Nonlinear Term $F = F(u, Du, D_x Du)$ on the Right-Hand Side Depends on u Explicitly	319
14.1	Introduction	319
14.2	Some Lemmas on Differential Inequalities	323
14.3	Upper Bound Estimates on the Life-Span of Classical Solutions to the Cauchy Problem of a Kind of Semi-linear Wave Equations—The Subcritical Case	327
14.4	Upper Bound Estimates on the Life-Span of Classical Solutions to the Cauchy Problem of a Kind of Semi-linear Wave Equations—The Critical Case	337
14.5	Proof of the Main Results	349
14.6	Appendix—Fuchs-Type Differential Equations and Hypergeometric Equations	353
14.6.1	Regular Singular Points of Second-Order Linear Ordinary Differential Equations	353
14.6.2	Fuchs-Type Differential Equations	356
14.6.3	Hypergeometric Equations	357
15	Applications and Developments	363
15.1	Applications	363

- 15.1.1 Potential Solutions to Compressible Euler Equations 363
 - 15.1.2 Time-Like Minimal Hypersurface in Minkowski Space 368
- 15.2 Some Further Results 369
 - 15.2.1 Further Results When $n = 2$ 369
 - 15.2.2 Further Results When $n = 3$ 370
- 15.3 Some Important Developments 371
 - 15.3.1 Three-Dimensional Nonlinear Elastodynamics Equations 371
 - 15.3.2 Einstein Equation in a Vacuum 375
- References** 383
- Index** 387