

Contents

1	Constructive Ideal Theory	1
1.1	Ideals and Gröbner Bases	2
1.1.1	Monomial Orderings	2
1.1.2	Gröbner Bases	4
1.1.3	Normal Forms	5
1.1.4	The Buchberger Algorithm	6
1.2	Elimination Ideals	8
1.2.1	Image Closure of Morphisms	9
1.2.2	Relations Between Polynomials	9
1.2.3	The Intersection of Ideals	10
1.2.4	The Colon Ideal	10
1.2.5	The Krull Dimension	11
1.3	Syzygy Modules	13
1.3.1	Computing Syzygies	13
1.3.2	Free Resolutions	16
1.4	Hilbert Series	17
1.4.1	Computation of Hilbert Series	20
1.5	The Radical Ideal	22
1.5.1	Reduction to Dimension Zero	22
1.5.2	Positive Characteristic	23
1.6	Normalization	24
	References	28
2	Invariant Theory	31
2.1	Invariant Rings	31
2.2	Reductive Groups	37
2.2.1	Linearly Reductive Groups	38
2.2.2	Other Notions of Reductivity	43
2.3	Categorical Quotients	45
2.4	Separating Invariants	48

2.5	Homogeneous Systems of Parameters	54
2.5.1	Hilbert's Nullcone.....	54
2.5.2	Existence of Homogeneous Systems of Parameters.....	56
2.6	The Cohen-Macaulay Property of Invariant Rings	57
2.6.1	The Cohen-Macaulay Property	57
2.6.2	The Hochster-Roberts Theorem	59
2.7	Hilbert Series of Invariant Rings	65
	References	67
3	Invariant Theory of Finite Groups	71
3.1	Homogeneous Components	72
3.1.1	The Linear Algebra Method	73
3.1.2	The Reynolds Operator	73
3.2	Noether's Degree Bound	74
3.3	Degree Bounds in the Modular Case	78
3.3.1	Richman's Lower Degree Bound.....	79
3.3.2	Symonds' Degree Bound	82
3.4	Molien's Formula	83
3.4.1	Characters and Molien's Formula	84
3.4.2	Extensions to the Modular Case.....	86
3.4.3	Extended Hilbert Series.....	89
3.5	Primary Invariants	91
3.5.1	Dade's Algorithm	92
3.5.2	An Algorithm for Optimal Homogeneous Systems Parameters	93
3.5.3	Constraints on the Degrees of Primary Invariants	94
3.6	Cohen-Macaulayness	97
3.7	Secondary Invariants	100
3.7.1	The Nonmodular Case	101
3.7.2	The Modular Case	104
3.8	Minimal Algebra Generators and Syzygies	106
3.8.1	Algebra Generators from Primary and Secondary Invariants	106
3.8.2	Direct Computation of Algebra Generators: King's Algorithm	107
3.8.3	Computing Syzygies	109
3.9	Properties of Invariant Rings	111
3.9.1	The Cohen-Macaulay Property	111
3.9.2	Free Resolutions and Depth	112
3.9.3	The Hilbert Series	115
3.9.4	Polynomial Invariant Rings and Reflection Groups	115
3.9.5	The Gorenstein Property	120

3.10	Permutation Groups	123
3.10.1	Direct Products of Symmetric Groups	123
3.10.2	Göbel's Algorithm	125
3.10.3	SAGBI Bases	130
3.11	Ad Hoc Methods	131
3.11.1	Finding Primary Invariants	132
3.11.2	Finding Secondary Invariants	134
3.11.3	The Other Exceptional Reflection Groups	138
3.12	Separating Invariants	139
3.12.1	Degree Bounds	139
3.12.2	Polynomial Separating Subalgebras and Reflection Groups	140
3.13	Actions on Finitely Generated Algebras	142
	References	147
4	Invariant Theory of Infinite Groups	153
4.1	Computing Invariants of Linearly Reductive Groups	153
4.1.1	The Heart of the Algorithm	153
4.1.2	The Input: The Group and the Representation	156
4.1.3	The Algorithm	159
4.2	Improvements and Generalizations	164
4.2.1	Localization of the Invariant Ring	165
4.2.2	Generalization to Arbitrary Graded Rings	169
4.2.3	Covariants	172
4.3	Invariants of Tori	174
4.4	Invariants of SL_n and GL_n	178
4.4.1	Binary Forms	180
4.5	The Reynolds Operator	182
4.5.1	The Dual Space $K[G]^*$	184
4.5.2	The Reynolds Operator for Semi-simple Groups	186
4.5.3	Cayley's Omega Process	193
4.6	Computing Hilbert Series	198
4.6.1	A Generalization of Molien's Formula	198
4.6.2	Hilbert Series of Invariant Rings of Tori	202
4.6.3	Hilbert Series of Invariant Rings of Connected Reductive Groups	204
4.6.4	Hilbert Series and the Residue Theorem	206
4.7	Degree Bounds for Invariants	216
4.7.1	Degree Bounds for Orbits	219
4.7.2	Degree Bounds for Tori	224
4.8	Properties of Invariant Rings	226
4.9	Computing Invariants of Reductive Groups	227
4.9.1	Computing Separating Invariants	228
4.9.2	Computing the Purely Inseparable Closure	232
4.9.3	Actions on Varieties	236

4.10	Invariant Fields and Localizations of Invariant Rings	240
4.10.1	Extendend Derksen Ideals and CAGEs	241
4.10.2	The Italian Problem	245
4.10.3	Geometric Aspects of Extended Derksen Ideals	246
4.10.4	Computational Aspects of Extended Derksen Ideals	248
4.10.5	The Additive Group	254
4.10.6	Invariant Rings and Quasi-affine Varieties	258
	References	261
5	Applications of Invariant Theory	265
5.1	Cohomology of Finite Groups	265
5.2	Galois Group Computation	266
5.2.1	Approximating Zeros	269
5.2.2	The Symbolic Approach	270
5.3	Noether's Problem and Generic Polynomials	272
5.4	Systems of Algebraic Equations with Symmetries	275
5.5	Graph Theory	276
5.6	Combinatorics	278
5.7	Coding Theory	281
5.8	Equivariant Dynamical Systems	283
5.9	Material Science	285
5.10	Computer Vision	288
5.10.1	View Invariants of 3D Objects	288
5.10.2	Invariants of n Points on a Plane	289
5.10.3	Moment Invariants	291
	References	293
A	Linear Algebraic Groups	297
A.1	Linear Algebraic Groups	297
A.2	The Lie Algebra of a Linear Algebraic Group	299
A.3	Reductive and Semi-simple Groups	303
A.4	Roots	304
A.5	Representation Theory	306
	References	307
B	Is One of the Two Orbits in the Closure of the Other?	309
B.1	Introduction	309
B.2	Examples	310
B.3	Algorithm	312
B.4	Defining the Set $\overline{G \cdot L}$ by Equations	317
	References	321
C	Stratification of the Nullcone	323
C.1	Introduction	323
C.2	The Stratification	325

C.3 The Algorithm 330

C.4 Examples 336

References 343

Addendum to Appendix C: The Source Code of HNC 345

References 358

Notation 359

Index 361